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Bernstein–Sato theory for linearly square-free polynomials in positive characteristic

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Resum *(CAT)*
La teoria de Bernstein–Sato ha esdevingut recentment un tema central a l'àlgebra commutativa i la geometria algebraica, atès que constitueix una poderosa eina per a classificar i quantificar singularitats en varietats algebraiques. En particular, ha sorgit un gran interès per estendre la teoria a anells de característica positiva. En aquest article, considerem una classe de polinomis, que denominem polinomis linealment lliures de quadrats, i investiguem els seus invariants associats en el context de la teoria de Bernstein–Sato.

Abstract *(ENG)*
Bernstein–Sato theory has recently emerged as a central topic in commutative algebra and algebraic geometry, as it constitutes a powerful tool in classifying and quantifying singularities of algebraic varieties. Notably, there has been a surge of interest in extending this theory to the positive characteristic setting. In this work, we consider a class of polynomials, which we call linearly square-free polynomials, and investigate their associated invariants within the context of Bernstein–Sato theory.

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1. Introduction

A central challenge driving the development of algebraic geometry is the classification of algebraic varieties, which includes the classification of the singularities of these varieties. One approach at tackling this problem is to characterize singularities by attaching algebraic invariants.

A rich family of such invariants falls under the umbrella of the so-called Bernstein–Sato theory, whose roots lie in the foundational works of Bernstein [2] and Sato [23]. We briefly outline their discovery. Denote by $\mathcal{D}_{R|\mathbb{C}}$ the ring of $\mathbb{C}$-linear differential operators on the polynomial ring $R = \mathbb{C}[x_1, \dots, x_n]$ and let f be a nonzero polynomial. Then there exist a nonzero differential operator $\delta(s) \in \mathcal{D}_{R|\mathbb{C}}[s]$, and a non-constant monic polynomial $b_f(s) \in \mathbb{C}[s]$ satisfying the functional equation

$$\delta(s) \cdot f^{s+1} = b_f(s)\, f^s, \quad \text{for } s \in \mathbb{Z}_{\geq 0}.$$

The polynomial $b_f(s)$ is the Bernstein–Sato polynomial of f.

The Bernstein–Sato polynomial has been the focus of extensive research since it encodes the behavior of the singularities of the hypersurface defined by f in $\mathbb{C}^n$. To showcase this, suppose f vanishes at $0 \in \mathbb{C}^n$. A well-known invariant from complex analysis is the log-canonical threshold of f at the origin, defined as

$$\operatorname{lct}(f) = \sup \left\{ \lambda \in \mathbb{R}_{>0} \;\middle|\; \int_U \frac{1}{|f|^{2\lambda}} < \infty \text{ for some neighborhood } U \text{ of the origin} \right\}.$$

The log-canonical threshold is a rational number in the interval $(0, 1]$. The more singular f is, the smaller the log-canonical threshold will be. Kollár proved that the log-canonical threshold of f is the smallest root of $b_f(-s)$ [13]. It is known that the roots of $b_f(s)$ are rational and negative due to Malgrange and Kashiwara [16, 12]. A number of invariants have originated around the Bernstein–Sato polynomial over the years. Of special interest in birational geometry are multiplier ideals and jumping numbers (for instance, see [15]).

In positive characteristic, Bernstein–Sato theory has a more recent development. Let us make an overview of one of the main objects of study, namely, the test ideals. These were introduced by Hochster and Huneke as an auxiliary tool in the context of tight closure theory [11], and afterwards related to the multiplier ideals by Hara and Yoshida [8]. Blickle, Mustaţă and Smith gave an alternative but equivalent definition of test ideals in [6], on which we base our study.

To fix ideas, let R be a regular ring of characteristic $p > 0$ and f a nonzero element. The test ideals (cf. Definition 2.10) are a family $\{\tau(f^\lambda)\}_{\lambda \in \mathbb{R}_{\geq 0}}$ of ideals of R indexed by the real numbers. For $\lambda \leq \mu$, these satisfy $\tau(f^\lambda) \supseteq \tau(f^\mu)$, hence one obtains a descending chain of ideals in R. One can show that for a fixed $\lambda > 0$, there exists $\varepsilon > 0$ such that $\tau(f^\lambda) = \tau(f^\mu)$ for all $\mu \in [\lambda, \lambda + \varepsilon)$, i.e. the family is right semi-continuous. On the contrary, there exist certain $\lambda > 0$ such that $\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda)$ for any $\varepsilon > 0$, that is, the chain of test ideals "jumps". These jumping spots are named F-jumping numbers (cf. Definition 2.14), and the smallest among them is the F-pure threshold, as introduced in [24]. Under finiteness hypotheses, F-jumping numbers are known to be discrete and rational (see Theorem 3.1 of [6]). Needless to say, these notions have been extended to non-principal ideals.

As the terminology suggests, the test ideals, F-jumping numbers and F-pure thresholds serve as characteristic $p > 0$ analogues to the multiplier ideals, jumping numbers and log-canonical thresholds, respectively. Remarkably, there is a deep and intricate relationship between these two theories. For instance, one can recover the log-canonical threshold from the F-pure threshold by letting $p \to \infty$ (see Theorem 3.4 in [19]). It is also known in several cases that the reduction modulo p of a multiplier ideal produces the corresponding test ideal [20].

The F-pure threshold has been computed in a handful of cases. It is known, for instance, in the case of elliptic curves, Calabi–Yau hypersurfaces, diagonal hypersurfaces and determinantal ideals, to name a few [3, 4, 10, 17]. Among the few situations where test ideals have been fully characterized, there is the case of determinantal ideals of maximal minors [9].

In general, finding F-jumping numbers and test ideals is a challenging problem, even in smooth ambient spaces such as polynomial rings and with the aid of computational tools. To some extent, the aforementioned known results rely on the favorable arithmetic and combinatorial properties of the objects involved. Without these properties, very little can be said about F-invariants.

Our goal in this article is to compute the F-jumping numbers and test ideals for a new class of polynomials, which we refer to as linearly square-free polynomials. These are polynomials whose monomials are all square-free, meaning they are not divisible by any square of an indeterminate. In the process, we also compute several other F-invariants useful for the theory, namely, the ν-invariants, Frobenius roots, and Bernstein–Sato roots, which we will introduce in due course. Finally, we relate these computations to the log-canonical threshold of linearly square-free polynomials in characteristic zero. This work originated from the study of F-invariants for determinants of generic matrices of indeterminates in characteristic $p > 0$. Subsequently, it was realized that the same ideas applied to linearly square-free polynomials.

Throughout, all rings considered will be commutative with unit.

2. Background

2.1 Frobenius powers and Frobenius roots

Let R be a ring of characteristic $p > 0$. We denote by $F \colon R \to R$, $f \mapsto f^p$ the Frobenius or p-th power map. This is a ring endomorphism of R. For an integer $e \geq 0$, we let $F^e \colon R \to R$, $f \mapsto f^{p^e}$ be the e-th iterate of the Frobenius.

Definition 2.1. For an integer $e \geq 0$, the e-th Frobenius power of an ideal $I \subseteq R$ is

$$I^{[p^e]} = F^e(I)R = (f^{p^e} \mid f \in I).$$

This is an ideal of R. In the case that I be generated by $f_1, \dots, f_n$, one has

$$I^{[p^e]} = (f_1^{p^e}, \dots, f_n^{p^e}).$$

Remark 2.2. When I is a principal ideal of R, say $I = (f)$, Frobenius powers and the usual powers coincide,

$$(f)^{[p^e]} = (f)^{p^e}.$$

A sort of converse operation to Frobenius powers are Frobenius roots. For principal ideals, Frobenius roots were first introduced in [1] by Àlvarez-Montaner, Blickle and Lyubeznik, in order to study generators of modules over rings of differential operators in positive characteristic. Afterwards, Frobenius roots were generalized to the non-principal case by Blickle, Mustață and Smith in [6].

Definition 2.3. For an integer $e \geq 0$, the e-th Frobenius root of an ideal $I \subseteq R$ is the smallest ideal $J \subseteq R$ in the sense of inclusion such that

$$I \subseteq J^{[p^e]}.$$

We denote the e-th Frobenius root of the ideal I by $I^{[1/p^e]}$. For $e = 0$, we set $I^{[1/p^e]} = I$.

A celebrated theorem of Kunz states that a ring R of characteristic $p > 0$ is regular if and only if the Frobenius $F\colon R \to R$ is a flat map [14]. Under the assumption of regularity, one can show that Frobenius roots are well-defined. See, for instance, Lemma 2.3 of [6].

Remark 2.4. Let I_1, I_2 be ideals of R such that $I_1 \subseteq I_2$. Then one has

$$I_1 \subseteq I_2 \subseteq (I_2^{[1/p^e]})^{[p^e]}.$$

Because $I_1^{[1/p^e]}$ is the smallest ideal with $I_1 \subseteq (I_1^{[1/p^e]})^{[p^e]}$, it follows that

$$I_1^{[1/p^e]} \subseteq I_2^{[1/p^e]},$$

hence Frobenius roots preserve ideal containments.

Remark 2.5. Let I, J be ideals of R and $e \geq 0$ an integer. Then

$$I \cdot J^{[1/p^e]} \subseteq (I^{[p^e]}J)^{[1/p^e]}.$$

Proposition 2.6 ([21, Lemma 2.3])**.** *Let I, J be ideals of R and $e \geq 0$ an integer. One has that $I^{[1/p^e]} \subseteq J$ if and only if $I \subseteq J^{[p^e]}$.*

We next describe a nice characterization of Frobenius roots in terms of generators, which will prove to be computationally useful. To this end, we endow R with an exotic R-module structure.

Definition 2.7. For an integer $e \geq 0$, define the R-module F^e_*R as follows. Its elements are denoted by F^e_*f, where f is in R. As an abelian group, F^e_*R is isomorphic to R, so addition is defined by

$$F^e_*f + F^e_*g = F^e_*(f+g), \quad \text{for } F^e_*f, F^e_*g \in F^e_*R.$$

The action of R on F^e_*R is defined by restricting scalars along the e-th iterate F^e of the Frobenius, that is,

$$r \cdot F^e_*f = F^e_*(r^{p^e}f), \quad \text{for } r \in R,\ F^e_*f \in F^e_*R.$$

Definition 2.8. A Noetherian ring R of characteristic $p > 0$ is an F-finite ring if F^e_*R is a finitely generated R-module for some $e \geq 1$ (equivalently, all $e \geq 1$).

Proposition 2.9 ([1, Section 3], [6, Proposition 2.5])**.** *Suppose that F^e_*R is a free R-module with basis $\varepsilon_1, \dots, \varepsilon_n$. Let I be an ideal of R generated by $f_1, \dots, f_m$. For a generator f_i, $i = 1, \dots, m$, write*

$$F^e_*f_i = g_{i,1}F^e_*\varepsilon_1 + \cdots + g_{i,n}F^e_*\varepsilon_n, \quad \text{where } g_{i,1}, \dots, g_{i,n} \in R.$$

Then the e-th Frobenius root of I is

$$I^{[1/p^e]} = (g_{i,j} \mid i = 1, \dots, m,\ j = 1, \dots, n).$$

2.2 Test ideals and ν-invariants

From now on, let R be a regular F-finite ring of characteristic $p > 0$. For a real number $x \in \mathbb{R}$, let $\lceil x \rceil \in \mathbb{Z}$ denote the round-up of x, i.e. the least integer greater or equal than x.

As mentioned earlier, the test ideals are the characteristic $p > 0$ analogues of the multiplier ideals. We adopt as a definition for the test ideal the characterization given in [6]:

Definition 2.10 ([6, Definition 2.9])**.** The test ideal of an ideal $I \subseteq R$ with exponent $\lambda \in \mathbb{R}_{\geq 0}$ is

$$\tau(I^\lambda) = \bigcup_{e=0}^{\infty} (I^{\lceil \lambda p^e \rceil})^{[1/p^e]}.$$

Remark 2.11. It can be shown that the ideals on the right-hand side give an ascending chain in R,

$$(I^{\lceil \lambda p \rceil})^{[1/p]} \subseteq (I^{\lceil \lambda p^2 \rceil})^{[1/p^2]} \subseteq \cdots \subseteq (I^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq (I^{\lceil \lambda p^{e+1} \rceil})^{[1/p^{e+1}]} \subseteq \cdots$$

(see Lemma 2.8 in [6]). Since R is a Noetherian ring, the chain eventually stabilizes:

$$\tau(I^\lambda) = (I^{\lceil \lambda p^e \rceil})^{[1/p^e]}, \quad \text{for some } e \gg 0.$$

Remark 2.12. Let $0 \leq \lambda \leq \mu$ be real numbers. Because $\lceil \lambda p^e \rceil \leq \lceil \mu p^e \rceil$, one has that

$$I^{\lceil \lambda p^e \rceil} \supseteq I^{\lceil \mu p^e \rceil}, \quad \text{for every } e \geq 1.$$

On the other hand, Remark 2.4 shows that Frobenius roots preserve inclusions, therefore

$$\tau(I^\lambda) \supseteq \tau(I^\mu), \quad \text{whenever } \mu \geq \lambda \geq 0.$$

It follows from the remark above that test ideals give a descending chain of ideals of R. More explicitly, given non-negative real numbers $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq$, one has that

$$\tau(I^{\lambda_1}) \supseteq \tau(I^{\lambda_2}) \supseteq \cdots \supseteq \tau(I^{\lambda_n}) \supseteq \cdots.$$

Such chain of ideals can "jump", i.e. the containments between test ideals may be strict. The results below encode this behavior:

Theorem 2.13 ([19, Remark 2.12], [6, Corollary 2.16, Theorem 3.1])**.** *Let I be an ideal of R.*

(i) For each $\lambda \geq 0$, there exists $\varepsilon > 0$ such that $\tau(I^\lambda) = \tau(I^{\lambda+\varepsilon})$. In particular, there exists $\lambda > 0$ small enough such that $\tau(I^\lambda) = R$.

(ii) There exist real numbers $\lambda > 0$ such that $\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda)$ for all $\varepsilon > 0$.

Definition 2.14 ([24, Definition 2.1], [19], [6, Definition 2.17])**.** Let I be an ideal of R. A real number $\lambda > 0$ is an F-jumping number of I if

$$\tau(I^{\lambda-\varepsilon}) \supsetneq \tau(I^\lambda), \quad \text{for every } \varepsilon > 0.$$

The smallest F-jumping number is called the F-pure threshold of I, and denoted by $\operatorname{fpt}(I)$, namely

$$\operatorname{fpt}(I) = \sup\{\lambda > 0 \mid \tau(I^\lambda) = R\}.$$

F-jumping numbers were introduced under the name F-thresholds in [19], as an invariant to study the jumping coefficients of the test ideals of Hara and Yoshida [8]. Afterwards, it was shown that the sets of F-thresholds and F-jumping numbers are equal (see [6, Corollary 2.30]). On another note, one has the following result relating the log-canonical threshold and the F-pure threshold:

Theorem 2.15 ([19, Theorem 3.4])**.** *Let f be a polynomial with integer coefficients in $\mathbb{C}[x_1, \dots, x_n]$. For a prime number $p > 0$, let f_p denote the reduction modulo p of f in $\mathbb{F}_p[x_1, \dots, x_n]$. Then*

$$\lim_{p \to \infty} \operatorname{fpt}(f_p) = \operatorname{lct}(f).$$

Another object closely related to the F-jumping numbers are the ν-invariants:

Definition 2.16 ([19])**.** Let I, J be ideals of R such that $I \subseteq \operatorname{rad} J$, where $\operatorname{rad} J$ denotes the radical of J. Fix an integer $e \geq 0$. The ν-invariant of level e of I with respect to J is

$$\nu_I^J(p^e) = \max\{r \in \mathbb{Z} \mid I^r \not\subseteq J^{[p^e]}\}.$$

Because $I \subseteq \operatorname{rad} J$, this integer exists and is finite. The set $\nu_I^\bullet(p^e)$ of ν-invariants of level e of I is the set of integers of the form $\nu_I^J(p^e)$ obtained as J ranges over the ideals containing I in its radical:

$$\nu_I^\bullet(p^e) = \{\nu_I^J(p^e) \mid J \subseteq R \text{ such that } I \subseteq \operatorname{rad} J\}.$$

Remark 2.17. In view of Proposition 2.6, $r \geq 0$ is the ν-invariant $\nu_I^J(p^e)$ if and only if $(I^r)^{[1/p^e]} \not\subseteq J$.

The ν-invariants were introduced precisely to study F-thresholds. In fact, the F-threshold $c^J(I)$ of I with respect to J was defined in [19] as

$$c^J(I) = \lim_{e \to \infty} \frac{\nu_I^J(p^e)}{p^e}.$$

Since F-thresholds and F-jumping numbers coincide when R is a regular ring, the ν-invariants are a powerful tool for shedding light on test ideals.

In computing the ν-invariants of an ideal I, it is not evident how to choose an ideal J that contains I in its radical. Instead, however, one can inspect the chain of ideals

$$\cdots \subseteq (I^{r+1})^{[1/p^e]} \subseteq (I^r)^{[1/p^e]} \subseteq \cdots \subseteq (I^2)^{[1/p^e]} \subseteq I^{[1/p^e]} \subseteq R.$$

In some cases, the containments are, in fact, equalities. When they are not, the chain of ideals "jumps". The next proposition, together with Remark 2.17, shows that the spot where the chain jumps is a ν-invariant.

Proposition 2.18 ([21, Proposition 4.2])**.** *The set of ν-invariants of level $e \geq 0$ of an ideal I is*

$$\nu_I^\bullet(p^e) = \{r \geq 0 \mid (I^{r+1})^{[1/p^e]} \neq (I^r)^{[1/p^e]}\}.$$

2.3 Bernstein–Sato roots

The last algebraic invariants relevant to our discussion are the Bernstein–Sato roots. These are characteristic $p > 0$ analogues to the roots of the Bernstein–Sato polynomial in characteristic zero, a concept that originated from Mustaţă's work [18]. Mustaţă initiated the extension of Bernstein–Sato polynomials to positive characteristic, an effort further advanced by Bitoun [5]. Due to the intricate nature of constructing Bernstein–Sato roots, we will instead use the more straightforward characterization in terms of ν-invariants, as provided by Quinlan-Gallego [21]. Before delving into this topic, we will briefly discuss p-adic limits and integers.

The p-adic valuation on $\mathbb{Z}$ is the map $v_p\colon \mathbb{Z} \to \mathbb{Z}_{\geq 0}$ defined by $v_p(0) = \infty$ and

$$v_p(n) = \max\{k \geq 0 \mid p^k \text{ divides } n\}, \quad \text{for } n \neq 0,$$

which naturally extends to a valuation $v_p\colon \mathbb{Q} \to \mathbb{Z}_{\geq 0}$ by letting

$$v_p\left(\frac{a}{b}\right) = v_p(a) - v_p(b).$$

This induces the p-adic norm $|\cdot|_p\colon \mathbb{Q} \to \mathbb{R}$, $|x|_p = p^{-v_p(x)}$, and in turn the p-adic metric $d_p\colon \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}$, $d_p(x, y) = p^{-v_p(x-y)}$. In this setting, the ring $\mathbb{Q}_p$ of p-adic numbers is the completion of $\mathbb{Q}$ with respect to the p-adic metric. The ring $\mathbb{Z}_p$ of p-adic integers is the subring of $\mathbb{Q}_p$ given by

$$\mathbb{Z}_p = \{\alpha \in \mathbb{Q}_p \mid |\alpha|_p \leq 1\}.$$

Because $v_p(n) \geq 0$ for every $n \in \mathbb{Z}$, one has $|n|_p \leq 1$, therefore $\mathbb{Z}$ is contained in $\mathbb{Z}_p$. From the definition, one also sees that $\mathbb{Q}$ is contained in $\mathbb{Q}_p$. A sequence $(x_n)_{n=0}^{\infty} \subseteq \mathbb{Q}$ has p-adic limit $\alpha \in \mathbb{Q}_p$ if $x_n \to \alpha$ in the p-adic metric. For more on p-adic numbers, we refer the interested reader to Section 7 in [21].

With this in mind, Bernstein–Sato roots are defined as follows:

Definition 2.19 ([21, Proposition 6.13], [22, Theorem IV.17])**.** Let I be an ideal of R. A p-adic integer $\alpha \in \mathbb{Z}_p$ is a Bernstein–Sato root of I if there exists a sequence $(\nu_e)_{e=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ of ν-invariants of I, $\nu_e \in \nu_I^{\bullet}(p^e)$, whose p-adic limit is α.

3. Linearly square-free polynomials

In this section we prove our main results, namely, the computation of Bernstein–Sato theory invariants for linearly square-free polynomials in characteristic $p > 0$.

Definition 3.1. Let $R = B[x_1, \dots, x_n]$ be a polynomial ring over a commutative ring B. We say that a polynomial in R is a linearly square-free polynomial if all its monomials are square-free.

Example 3.2. Let $R = B[x_{11}, \dots, x_{1n}, \dots, x_{n1}, \dots, x_{nn}]$ be a polynomial ring in n^2 indeterminates. The indeterminates may be assembled in an $n \times n$ generic matrix of indeterminates $X = (x_{ij})$. Then the determinant of X,

$$\det X = \sum_{\sigma \in \mathrm{Sym}(n)} \mathrm{sgn}(\sigma) x_{1\sigma(1)} \cdots x_{n\sigma(n)},$$

is a linearly square-free polynomial.

Example 3.3. Let $X = (x_{ij})$ be a $2n \times 2n$ skew-symmetric matrix of indeterminates, that is, $x_{ij} = -x_{ji}$ for $1 \leq i, j \leq 2n$. The Pfaffian of X is the polynomial

$$\operatorname{Pf} X = \frac{1}{2^n n!} \sum_{\sigma \in \operatorname{Sym}(2n)} \operatorname{sgn}(\sigma) \prod_{i=1}^{n} x_{\sigma(2i-1)\sigma(2i)}.$$

It can be shown that the Pfaffian satisfies $(\operatorname{Pf} X)^2 = \det X$. Since no indeterminate appears twice in the same monomial, the Pfaffian is linearly square-free.

Example 3.4. Let K be a field and $W \subseteq K^E$ be a realization of a matroid M, where E is a finite set that forms a basis of K^E. Then the configuration polynomial of W is linearly square-free (see [7]). These polynomials have applications in physics.

The proposition below is a well-known fact that shows that $F^e_* R$ has a particularly nice structure provided R is a polynomial ring over a perfect field of characteristic $p > 0$. Recall that a field K of characteristic $p > 0$ is perfect if the Frobenius $F \colon K \to K$ is an automorphism of K. This is tantamount to every element of K having a p^e-th root in K.

Proposition 3.5. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic* $\operatorname{Char}(K) = p > 0$*. For each integer $e \geq 0$, one has that*

$$F^e_* R \simeq \bigoplus_{0 \leq i_1, \dots, i_n < p^e} R\, F^e_* x_1^{i_1} \cdots x_n^{i_n}.$$

In consequence, the set $\{F^e_* x_1^{i_1} \cdots x_n^{i_n} \mid 0 \leq i_1, \dots, i_n < p^e\}$ is a basis for $F^e_* R$. We refer to this as the standard basis of $F^e_* R$.

We start by computing the Frobenius roots and the ν-invariants of linearly square-free polynomials. This will lay the groundwork for further results. For the following lemma, it will be convenient to use multi-index notation. If $B[x_1, \dots, x_n]$ is a polynomial ring in n variables, and $a = (a_1, \dots, a_n) \in \mathbb{Z}^n_{\geq 0}$ is an n-tuple of non-negative integers, we let

$$x^a = x_1^{a_1} \cdots x_n^{a_n}.$$

Lemma 3.6. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. Let f be a linearly square-free polynomial. Fix an integer $e \geq 0$. Then for all integers $0 \leq r < p^e$, $F^e_* f^r$ is a nonzero K-linear combination of elements in the standard basis of $F^e_* R$.*

Proof. Because f is linearly square-free, one has

$$f = \sum_{i=1}^{m} \alpha_i x^{a_i}, \quad \text{where } \alpha_i \in K,\ a_i = (a_{i1}, \dots, a_{in}) \in \{0,1\}^n,$$

for some integer $m \geq 1$, therefore

$$f^r = \sum_{k_1 + \cdots + k_m = r} \binom{r}{k_1, \dots, k_m} \prod_{i=1}^{m} \alpha_i^{k_i} x^{k_i a_i}.$$

The monomials in the expression above have the form

$$\prod_{i=1}^{m} x^{k_i a_i} = x^{\sum_{i=1}^m k_i a_i} = x_1^{\sum_{i=1}^m k_i a_{i1}} \cdots x_n^{\sum_{i=1}^m k_i a_{in}}.$$

By assumption $0 \leq r < p^e$, hence the indeterminate x_j appears in each monomial with exponent

$$\sum_{i=1}^{m} k_i a_{ij} \leq \sum_{i=1}^{m} k_i = r < p^e.$$

It follows that

$$F_*^e x^{\sum_{i=1}^m k_i a_i}, \quad \text{for } i = 1, \dots, m,$$

is an element in the standard basis of $F_*^e R$. As a result, up to collecting terms, $F_*^e f^r$ reads

$$F_*^e f^r = \sum_{k_1+\cdots+k_m=r} \left(\binom{r}{k_1, \dots, k_m} \prod_{i=1}^{m} \alpha_i^{k_i} \right)^{1/p^e} F_*^e x^{\sum_{i=1}^m k_i a_i},$$

which proves that the coefficients are in K. Because $f^r \neq 0$ and $F_*^e R$ is a free R-module, some coefficient is nonzero. □

Theorem 3.7. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. Let f be a linearly square-free polynomial. Fix an integer $e \geq 0$. Then:*

(i) For all integers $s \geq 0$ and $0 \leq r < p^e$,

$$(f^{sp^e+p^e-1})^{[1/p^e]} = (f^{sp^e+p^e-2})^{[1/p^e]} = \cdots = (f^{sp^e+1})^{[1/p^e]} = (f^{sp^e})^{[1/p^e]} = (f)^s.$$

(ii) The ν-invariants of f of level e are $\nu_f^\bullet(p^e) = \{(s+1)p^e - 1 \mid s \in \mathbb{Z}_{\geq 0}\}$.

(iii) If $s \geq 0$ is an integer and $J = (f)^{s+1}$, then $\nu_f^J(p^e) = (s+1)p^e - 1$.

Proof. (i) For a fixed integer $s \geq 0$, Frobenius roots give an ascending chain

$$(f^{sp^e+p^e-1})^{[1/p^e]} \subseteq (f^{sp^e+p^e-2})^{[1/p^e]} \subseteq \cdots \subseteq (f^{sp^e+1})^{[1/p^e]} \subseteq (f^{sp^e})^{[1/p^e]}.$$

In the case $s = 0$, Lemma 3.6 shows that $F_*^e f^{p^e-1}$ is a nonzero K-linear combination of elements in the standard basis of $F_*^e R$. It follows from Proposition 2.9 that the Frobenius root $(f^{p^e-1})^{[1/p^e]}$ is generated by units of R, therefore $(f^{p^e-1})^{[1/p^e]} = R$. Now suppose that $s \geq 1$. In view of the ascending chain above, to prove equality it suffices to verify that

$$(f)^s \subseteq (f^{sp^e+p^e-1})^{[1/p^e]} \quad \text{and} \quad (f^{sp^e})^{[1/p^e]} \subseteq (f)^s.$$

On the one hand, by Remark 2.5,

$$(f)^s = (f)^s \, (f^{p^e-1})^{[1/p^e]} = (f^{s[p^e]} f^{p^e-1})^{[1/p^e]} = (f^{sp^e+p^e-1})^{[1/p^e]}.$$

On the other hand, by Proposition 2.6, the containment $(f^{sp^e})^{[1/p^e]} \subseteq (f)^s$ is equivalent to $(f)^{sp^e} \subseteq (f)^{s[p^e]} = (f)^{sp^e}$.

(ii) Part (i) shows that for each integer $s \geq 0$,

$$(f)^{s+1} = (f^{(s+1)p^e})^{[1/p^e]} \subsetneq (f^{(s+1)p^e-1})^{[1/p^e]} = (f)^s.$$

Then by Proposition 2.18, the ν-invariants of f of level $e \geq 0$ are of the form $(s+1)p^e - 1$ for $s \in \mathbb{Z}_{\geq 0}$.

(iii) It follows at once from Definition 2.16 and part (ii). □

Lemma 3.8. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$ and f be a linearly square-free polynomial. Let $\lambda \geq 0$ be a real number and $e \geq 0$ an integer. Then*

$$(f^{\lceil \lambda p^e \rceil})^{[1/p^e]} = \begin{cases} (f)^{\lfloor \lambda \rfloor} & \text{if } \{\lambda\} \leq (p^e-1)/p^e, \\ (f)^{\lfloor \lambda \rfloor + 1} & \text{if } \{\lambda\} > (p^e-1)/p^e, \end{cases}$$

where $\{\lambda\}$ denotes the fractional part of λ.

Proof. Write λ as $\lambda = \lfloor \lambda \rfloor + \{\lambda\}$. If $\{\lambda\} \leq (p^e-1)/p^e$, one has that $\lfloor \lambda \rfloor p^e \leq \lambda p^e \leq \lfloor \lambda \rfloor p^e + p^e - 1$, therefore $\lfloor \lambda \rfloor p^e \leq \lceil \lambda p^e \rceil \leq \lfloor \lambda \rfloor p^e + p^e - 1$. Theorem 3.7 shows

$$(f)^{\lfloor \lambda \rfloor} = (f^{\lfloor \lambda \rfloor p^e + p^e - 1})^{[1/p^e]} \subseteq (f^{\lceil \lambda p^e \rceil})^{[1/p^e]} \subseteq (f^{\lfloor \lambda \rfloor p^e})^{[1/p^e]} \subseteq (f)^{\lfloor \lambda \rfloor}.$$

On the other hand, suppose that $\{\lambda\} > (p^e-1)/p^e$. Similarly, one finds $\lfloor \lambda \rfloor p^e + p^e - 1 < \lambda p^e < \lfloor \lambda \rfloor p^e + p^e$, which gives $\lceil \lambda p^e \rceil = \lfloor \lambda \rfloor p^e + p^e$. Again using Theorem 3.7 gives

$$(f^{\lceil \lambda p^e \rceil})^{[1/p^e]} = (f^{(\lfloor \lambda \rfloor + 1)p^e})^{[1/p^e]} = (f)^{\lfloor \lambda \rfloor + 1},$$

thus proving the lemma. □

Theorem 3.9. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. Let f be a linearly square-free polynomial. Then:*

(i) For a real number $\lambda \geq 0$, one has $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor}$.

(ii) The set of F-jumping numbers of f is $\mathrm{FJN}(f) = \mathbb{Z}_{\geq 1}$. In particular, the F-pure threshold of f is 1.

Proof. (i) Since the sequence $((p^e-1)/p^e)_{e=0}^{\infty}$ has limit 1 as $e \to \infty$, there is an integer e_0 satisfying $\{\lambda\} \leq (p^e-1)/p^e$ for all $e \geq e_0$. It follows from Lemma 3.8 that $(f^{\lceil \lambda p^e \rceil})^{[1/p^e]} = (f)^{\lfloor \lambda \rfloor}$ for $e \geq e_0$, therefore $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor}$.

(ii) Fix an integer $n \geq 0$. Then $\tau(f^\lambda) = (f)^{\lfloor \lambda \rfloor} = (f)^n$ for all real numbers λ with $n \leq \lambda < n+1$. On the other hand, one has $\tau(f^{n+1}) = (f)^{n+1}$. Consequently $n+1$ is an F-jumping number of f, and the assertion follows. □

Corollary 3.10. *Let f be a linearly square-free polynomial with integer coefficients in $\mathbb{C}[x_1, \dots, x_n]$. The log-canonical threshold of f is $\mathrm{lct}(f) = 1$.*

Proof. Let $p > 0$ be a prime number and f_p be the reduction modulo p of f in $\mathbb{F}_p[x_1, \dots, x_n]$. If p does not divide all the coefficients of f, then f_p is nonzero and thus linearly square-free, hence $\mathrm{fpt}(f_p) = 1$ by Lemma 3.8. This occurs for all p large enough, so $\mathrm{lct}(f) = 1$ by Theorem 2.15. □

Corollary 3.11. *Let $R = K[x_1, \dots, x_n]$ be a polynomial ring over a perfect field K of characteristic $p > 0$. The only Bernstein–Sato root of a linearly square-free polynomial f is $\alpha = -\operatorname{fpt}(f) = -1$.*

Proof. Let $(t_d)_{d=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ be a sequence of non-negative integers and define

$$\nu_d := (t_d + 1)p^d - 1, \quad \text{for } d \geq 0.$$

In view of Theorem 3.7, each ν_d is a ν-invariant of f. We thus obtain a sequence $(\nu_d)_{d=0}^{\infty} \subseteq \mathbb{Z}_{\geq 0}$ of ν-invariants with p-adic limit $\nu_d \to \alpha = -1$ as $d \to \infty$. In consequence, $\alpha = -\operatorname{fpt}(f)$ is a Bernstein–Sato root of f. Because any sequence of ν-invariants of f is of this form, it follows that $\alpha = \operatorname{fpt}(f)$ is the only Bernstein–Sato root of f. □

The corollary above allows one to answer the following question.

Question 3.12 ([21, Question 6.16])**.** *Suppose that the F-pure threshold α of an ideal I lies in $\mathbb{Z}_{(p)}$, the localization of $\mathbb{Z}$ at $\{p^k \mid k \geq 0\}$. Is the largest Bernstein–Sato root of I equal to $-\alpha$?*

The answer is affirmative for linearly square-free polynomials in any characteristic $p > 0$.

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